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Random Integral Equation Formulation of a Generalized Langevin Equation

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In this paper, the generalized Langevin equation introduced by Kubo and Mori is formulated as a random integral equation. We consider (1) the existence and uniqueness of the solution, (2) moments of the solution process, (3) a comparison theorem for solution processes, and (4) the Cauchy polygonal approximation to the solution.

KEY WORDS: Brownian motion; Langevin equation; random integral equation; Banach fixed point theorem; Cauchy polygonal approximation method.

1. INTRODUCTION

In the classical theory of Brownian motion, say in the case of a *free* particle, one starts from the Langevin equation

$$\dot{u}(t) = -\beta u + R(t, \omega) \tag{1}$$

where $-\beta u$ represents the average force from the environment acting on the particle and giving rise to viscosity or friction, and $R(t, \omega)$ takes into account the extremely rapidly varying part of the force, bearing in mind the almost

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continuous individual impacts of liquid or gas molecules with the Brownian particle. {As a concrete example, one can consider a simple passive electrical circuit subject to spontaneous fluctuations of electric current $i(t)$. In this case, Eq. (1) becomes

$$L[di(t)/dt] + Ri(t) = E(t)$$

where $E(t)$ is a fluctuating voltage term whose timescale is very rapid compared to the relaxation time L/R.} The random force $R(t, \omega)$ in Eq. (1) is assumed to satisfy the following conditions: (a) $R(t, \omega)$ is independent of u; (b) $R(t, \omega)$ is centered and Gaussian; and

$$\text{(c)} \qquad \mathscr{E}\{R(s)R(t)\} = \sigma\delta(s - t) \tag{2}$$

In the above, $\mathscr{E}$ is the expectation operator and δ is the Dirac delta function. These conditions are reasonable; for example, the Gaussian assumption is quite reasonable for a Brownian particle having a mass much larger than the colliding molecules, since the motion is a result of numerous successive independent collisions, which enables one to appeal to the central limit theorem. For a Brownian particle under an external force field (for example, a harmonic oscillator), the Langevin equation takes the form

$$\dot{u}(t) = -\beta u + A(t) + R(t, \omega) \tag{3}$$

where $A(t)$ is the external force and the random force $R(t, \omega)$ satisfies the same conditions as those of a free particle. For detailed treatments of the probabilistic theory of Brownian motion, we refer to the books of Doob[1] Itô and McKean,[2] and Skorohod.[3]

For physical reasons, the idealizations in the classical theory need to be modified for a more realistic treatment. Kubo[4] and Mori[5] have considered a generalization of Brownian motion theory to random motion of a particle which need *not* necessarily be heavier than the interacting molecules of the media. Thus the time scale of the molecular motion is no longer shorter than that of the particle under motion and this forces us to drop assumption (c). By considering more realistic situations which generalized Brownian movement, Kubo and Mori were led to a natural extension of the Langevin equation in the form

$$\dot{u}(t) = -\int_{t_0}^{t} \gamma(t - \tau)\, u(\tau)\, d\tau + A(t) + R(t, \omega), \qquad t > t_0 \tag{4}$$

where $\gamma(t)$ is a retarded effect of the (time-dependent) frictional force, $A(t)$ is an external force, and $R(t, \omega)$ is the random force not correlated with the initial velocity $u(t_0)$.

In passing from the motion of Newtonian particles to Brownian particles, one goes from reversible processes to irreversible processes, as can easily be seen by computing the mean-square displacement of the particles. A basis for the study of irreversible processes is provided by finding closed formulas for the admittances to mechanical perturbations and kinetic coefficients in non-equilibrium system in terms of time correlations of physical quantities. From statistical mechanical considerations, Kubo (as well as Mori) established these fluctuation-dissipation relations, thereby extending the Nyquist theorem.

In the mathematical treatment of the classical Langevin equation, the Langevin equation is replaced by a formal differential equation for the velocity process $u(t, \omega)$. In order to give a correct stochastic analog of the Langevin equation, it is necessary to remember that $\dot{u}(t)$ need not exist, for example, the Brownian paths are not differentiable. The Langevin equation is rewritten in the following form:

$$du(t) = -\beta u(t)\, dt + A(t)\, dt + d\beta(t) \tag{5}$$

A rigorous interpretation of this differential equation is obtained, via stochastic integrals, by writing it in the integrated form

$$u(t) = u(t_0) - \int_{t_0}^{t} \beta u(s)\, ds + \int_{t_0}^{t} A(s)\, ds + \beta(t) - \beta(t_0) \tag{6}$$

Equation (6) is a special case of the more general Itô integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} m(s, x(\tau))\, d\tau + \int_{t_0}^{t} \sigma(\tau, x(\tau))\, d\beta(\tau) \tag{7}$$

[The last integral in (7) is a stochastic integral. For treatments of stochastic integrals, we refer to the books of Bharucha-Reid,(6) Doob,(1) and Skorohod.(3)]

In this paper, we give a rigorous treatment of the generalized Langevin equation. We take as the correct formulation of the generalized Langevin equation the following random integral equation:

$$\begin{aligned} x(t, \omega) = x(t_0, \omega) &+ \int_{t_0}^{t} \int_{t_0}^{s} \gamma(s - u)\, x(u, \omega)\, du\, ds \\ &+ \int_{t_0}^{t} m(u, x(u, \omega))\, du + \int_{t_0}^{t} \sigma(u, x(u, \omega))\, dM(u, \omega) \end{aligned} \tag{8}$$

In Eq. (7), $\beta(t, \omega)$ is a Brownian motion or Wiener process. Corresponding to (2), $\beta(t)$ satisfies the condition

$$\mathscr{E}\{| \beta(t, \omega) - \beta(s, \omega)|^2\} = \sigma^2 | t - s | \tag{9}$$

where σ^2 is a positive constant. In generalizing the Langevin equation, we have to abandon condition (2). In Eq. (8), the stochastic process $M(t, \omega)$ is not a Brownian motion process, but is assumed to be a continuous martingale. Thus the stochastic integral in (8) is the *Itô–Doob integral.*

In place of (9), $M(t, \omega)$ satisfies the following condition: $\{M(t), \mathscr{F}_t, t \in [t_0, T]\}$ is a continuous martingale such that there is a nondecreasing function $F(t)$, $t \in [t_0, T]$, with the property that, for $s < t$,

$$\mathscr{E}\{[M(t, \omega) - M(s, \omega)]^2 | \mathscr{F}_s\} = F(t) - F(s) \tag{10}$$

where $\mathscr{E}\{\cdot \mid \mathscr{F}_t\}$ is the conditional expectation relative to the σ-algebra $\mathscr{F}_t$.

We remark that a function $F(t)$ with the above property exists if $M(t, \omega)$ is a stochastic process with orthogonal increments (cf. Doob[11]).

In Section 2, we prove the existence and uniqueness of the solution of Eq. (8). The moments of the solution process $x(t, \omega)$ are considered in Section 3. In Section 4, we prove a comparison theorem for the solution process. Finally, in Section 5, we consider the ploygonal approximation of the solution.

2. EXISTENCE AND UNIQUENESS OF THE SOLUTION

Let $(\Omega, \mathscr{F}, \mu)$ be a fixed, complete probability space, and let $[t_0, T]$ be a fixed interval on the real line. $\{\mathscr{F}_t, t \in [t_0, T]\}$ is an increasing family of sub-σ-algebras of $\mathscr{F}$, and each $\mathscr{F}_t$ is complete with respect to the probability measure μ. In writing the stochastic process $\{x(t, \omega), t \in [t_0, T]\}$ we will suppress the argument ω and write it as $\{x(t), t \in [t \in [t_0, T]\}$. Every random variable x stands for its equivalence class. All the stochastic processes considered in this paper can and will be assumed separable.

In this section, we establish the existence and uniqueness of the solution of the random integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} \int_{t_0}^{s} \gamma(s - u)\, x(u)\, du\, ds + \int_{t_0}^{t} m(u, x(u))\, du + \int_{t_0}^{t} \sigma(u, x(u))\, dM(u) \tag{11}$$

Equation (11) is to be solved for $t \in (t_0, T]$. We make the following assumptions:

(A) $\{M(t), \mathscr{F}_t, t \in [t_0, T]\}$ is a continuous martingale such that there exists a nondecreasing function $F(t)$, $t \in [t_0, T]$, with the property that, for $s < t$,

$$\mathscr{E}\{[M(t) - M(s)]^2 \mid \mathscr{F}_s\} = F(t) - F(s) \tag{12}$$

(B) The functions $m(t, x)$ and $\sigma(t, x)$ are measurable in the pair (t, x), for $t_0 \leqslant t \leqslant T$ and $-\infty < x < \infty$.

(C) For each $t \in (t_0, T)$,

$$\int_{t_0}^{t} \mathscr{E}\{| m(\tau, x)|^2 + | \sigma(\tau, x)|^2\} \, dF(\tau) < \infty \tag{13}$$

We remark that under condition (13), the Itô–Doob integral is defined. Also, the stochastic integral

$$\int_{t_0}^{t} \xi(t, \omega) \, dM(t, \omega)$$

is defined for the class of functions $\xi(t, \omega)$ such that

$$\mu\left(\left\{\int_{t_0}^{T} | \xi(t)|^2 \, dF(t) < \infty\right\}\right) = 1$$

We define the convolution $(\gamma * \mathrm{x})$ as follows: Let $\gamma(\cdot) \in L_1([t_0, T])$ and $x \in L_\infty([t_0, T])$; then, for $s \in [t_0, T]$,

$$(\gamma * x)(s) = \int_{t_0}^{s} \gamma(s - u) \, x(u) \, du, \quad \text{a.s.} \tag{14}$$

It is well known that $(\gamma * \mathrm{x})$ is uniformly continuous on $[t_0, T]$. Using (14), Eq. (11) can be rewritten as

$$x(t) = x_0 + \int_{t_0}^{t} (\gamma * x)(u) \, du + \int_{t_0}^{t} m(u, x(u)) \, du + \int_{t_0}^{t} \sigma(u, x(u)) \, dM(u) \tag{15}$$

We now establish the existence and uniqueness of the solution of the random integral equation (15). We need the following lemmas.

Lemma 2.1. Let $\varphi(\cdot)$ be the mapping on $L_\infty([t_0, T])$ defined by

$$\varphi[x(t)] = x_0 + \int_{t_0}^{t} (\gamma * x)(u) \, du + \int_{t_0}^{t} m(u, x(u)) \, du + \int_{t_0}^{t} \sigma(u, x(u)) \, dM(u) \tag{16}$$

Let the following hold:

(1) $m(t, x)$ and $\sigma(t, x)$ are measurable in the pair (t, x) for $t \in [t_0, T]$ and $-\infty < x < \infty$.

(2) (Growth condition). There is a constant k such that for $t \in [t_0, T]$ and every x,

$$
\begin{aligned}
|m(t, x)|^2 &\leqslant k^2(1 + x^2) \\
|\sigma(t, x)|^2 &\leqslant k^2(1 + x^2)
\end{aligned}
\tag{17}
$$

Then φ is a mapping into $L_\infty([t_0, T])$, i.e., $\varphi\colon L_\infty \to L_\infty$.

Proof. From (17), we have

$$
\begin{aligned}
\| m(\cdot, x(\cdot))\|_\infty^2 &= \operatorname*{ess\,sup}_{t\in[t_0,T]} | m(t, x(t))|^2 \\
&\leqslant k^2(1 + \operatorname*{ess\,sup}_{t\in[t_0,T]} | x(t)|^2) \\
&= k^2(1 + \| x \|_\infty^2) < \infty
\end{aligned}
\tag{18}
$$

Thus $m(\cdot, x(\cdot)) \in L_\infty([t_0, T])$. Similarly, $\sigma(\cdot, x(\cdot)) \in L_\infty([t_0, T])$. From the uniform continuity of $(\gamma * \text{x})$ on $[t_0, T]$, there exists an $M_0 > 0$ such that $|\gamma * x| \leqslant M_0$. Now,

$$
\begin{aligned}
\left|\int_s^t (\gamma * x)(u)\, du\right|^2 &\leqslant (t - s) \int_s^t |(\gamma * x)(u)|^2\, du \\
&\leqslant M_0^2(t - s)^2
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\left|\int_s^t m(u, x(u))\, du\right|^2 &\leqslant (t - s) \int_s^t | m(u, x(u))|^2\, du \\
&\leqslant (t - s)^2 \| m(\cdot, x(\cdot))\|_\infty^2 \\
&\leqslant k^2(t - s)^2 (1 + \| x \|_\infty^2) < \infty
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
\left|\int_s^t \sigma(u, x(u))\, dM(u)\right|^2 &\leqslant \int_s^t | \sigma(u, x(u))|^2\, dF(u) \\
&\leqslant \| \sigma(\cdot, x(\cdot))\|_\infty^2 [F(t) - F(s)] \\
&\leqslant k^2(1 + \| x \|_\infty^2) \bigvee_{t_0}^{T} F < \infty
\end{aligned}
\tag{21}
$$

where $\vee_{t_0}^T F$ denotes the total variation of F in $[t_0, T]$ (F is a monotone function and hence is a function of bounded variation). From (19), (20), and (21),

$$\begin{aligned}\| \varphi(x)\|_\infty^2 &= \operatorname*{ess\,sup}_{t\in[t_0,T]} | \varphi(x)|^2 \\ &\leqslant 4 \| x_0 \|_\infty^2 + 4M_0^2(T - t_0)^2 \\ &\quad + 4k^2(1 + \| x \|_\infty^2)\Big[(T - t_0)^2 + \bigvee_{t_0}^{T} F\Big] < \infty \end{aligned} \tag{22}$$

Thus φ: $L_\infty([t_0, T]) \to L_\infty([t_0, T]$, which proves the lemma.

Lemma 2.2. Let φ be the mapping defined by (16), and let the functions $m(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ satisfy conditions (1) and (2) of Lemma 2.1. Let $m(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ also satisfy the following uniform Lipschitz condition:

(3) There is a constant k such that for $t \in [t_0, T]$ and x and y

$$\begin{aligned} | m(t, x) - m(t, y)| &\leqslant k \mid x - y \mid \\ | \sigma(t, x) - \sigma(t, y)| &\leqslant k \mid x - y \mid \end{aligned} \tag{23}$$

Then, some power of φ: $L_\infty \to L_\infty$ is a contraction.

Proof. For $\xi, \eta \in L_\infty$, define

$$\begin{aligned} \Delta_i(\cdot) &= \varphi^i\xi(\cdot) - \varphi^i\eta(\cdot) \\ \Delta m_i(\cdot) &= m(\cdot, \varphi^i\xi(\cdot)) - m(\cdot, \varphi^i\eta(\cdot)) \\ \Delta\sigma_i(\cdot) &= \sigma(\cdot, \varphi^i\xi(\cdot)) - \sigma(\cdot, \varphi^i\eta(\cdot)) \end{aligned}$$

Then,

$$\begin{aligned} | \Delta_i(t)|^2 &= \Big|\int_{t_0}^t (\gamma * \varphi^{i-1}\xi - \gamma * \varphi^{i-1}\eta)\, d\tau + \int_{t_0}^t \Delta m_i(\tau)\, d\tau \\ &\quad + \int_{t_0}^t \Delta\sigma_i(\tau)\, dM(\tau)\Big|^2 \\ &\leqslant 3\Big|\int_t^t [\gamma * (\varphi^{i-1}\xi - \varphi^{i-1}\eta)](\tau)\, d\tau\Big|^2 \\ &\quad + 3\Big|\int_{t_0}^t \Delta m_i(\tau)\, d\tau\Big|^2 + 3\Big|\int_{t_0}^t \Delta\sigma_i(\tau)\, dM(\tau)\Big| \\ &\leqslant 3(T - t_0)\| \gamma \|_1^2 \int_{t_0}^t | \Delta_{i-1} |^2\, d\tau + 3(T - t_0)^2 k^2 \int_{t_0}^t | \Delta_{i-1} |^2\, d\tau \\ &\quad + 3k^2 \int_{t_0}^t | \Delta_{i-1} |^2\, dF(\tau) \end{aligned} \tag{24}$$

Put $L = (T - t_0)$, $A = 3L \| \gamma \|_1^2 + 3k^2$, and $B = 3k^2$. Then, (24) becomes

$$| \Delta_i(t)|^2 \leqslant A \int_{t_0}^{t} | \Delta_{i-1}(\tau)|^2 \, d\tau + B \int_{t_0}^{t} | \Delta_{i-1}(\tau)|^2 \, dF(\tau) \tag{25}$$

In particular, for $i = 1$ and 2, we have

$$| \Delta_1(t)|^2 \leqslant A \| \Delta_0 \|_\infty^2 \int_{t_0}^{t} d\tau_1 + B \| \Delta_0 \|_\infty^2 \int_{t_0}^{t} dF(\tau_1) \tag{26}$$

and

$$\begin{aligned}
| \Delta_2(t)|^2 &\leqslant A \int_{t_0}^{t} | \Delta_1(\tau)|^2 \, d\tau + B \int_{t_0}^{t} | \Delta_1(\tau)|^2 \, dF(\tau) \\
&\leqslant A \int_{t_0}^{\tau_1} \Big[A \| \Delta_0 \|_\infty^2 \int_{t_0}^{t} d\tau_1 + B \| \Delta_0 \|_\infty^2 \int_{t_0}^{t} dF(\tau_1)\Big] \, d\tau \\
&\quad + B \int_{t_0}^{\tau_1} \Big[A \| \Delta_0 \|_\infty^2 \int_{t_0}^{t} d\tau_1 + B \| \Delta_0 \|_\infty^2 \int_{t_0}^{t} dF(\tau)\Big] \, d\tau_1 \\
&= \| \Delta_0 \|_\infty^2 \Big[A^2 \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 + AB \int_{t_0}^{t} dF(\tau_1) \int_{t_0}^{\tau_1} d\tau_2 \\
&\quad + AB \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} dF(\tau_2) + B^2 \int_{t_0}^{t} dF(\tau_1) \int_{t_0}^{\tau_1} dF(\tau_2)\Big] \\
&= \| \Delta_0 \|_\infty^2 \Big([A^2(t - t_0)^2/2!] + AB \Big\{(\tau_1 - t_0) \, F(\tau_1)\big|_{t_0}^{t} \\
&\quad - \int_{t_0}^{t} F(\tau_1) \, d\tau_1 + \int_{t_0}^{t} [F(\tau_1) - F(t_0)] \, d\tau_1\Big\} + (B^2/2!)\Big(\bigvee_{t_0}^{t} F\Big)^2\Big) \\
&\leqslant \| \Delta_0 \|_\infty^2 \, ((A^2L^2/2!) + AB\{(t - t_0)[F(t) - F(t_0)]\} + (B^2V^2/2!)) \\
&\leqslant \| \Delta_0 \|_\infty^2 \, [(A^2L^2/2!) + (2ABLV/2!) + (B^2V^2/2!)] \\
&= \| \Delta_0 \|_\infty^2 \, (AL + BV)^2/2
\end{aligned} \tag{27}$$

where $V = \bigvee_{t_0}^{T} F = F(T) - F(t_0)$. Thus from (26) and (27), we have

$$\| \Delta_1 \|_\infty^2 \leqslant \| \Delta_0 \|_\infty^2 \, (AL + BV)$$

$$\| \Delta_2 \|_\infty^2 \leqslant \| \Delta_0 \|_\infty^2 \, (AL + BV)^2/2!$$

We now show by induction that

$$\| \Delta_m \|_\infty^2 \leqslant \| \Delta_0 \|_\infty^2 \, (AL + BV)^m/m! \tag{28}$$

As we have seen, (28) is true for $m = 1$ and 2. Let (28) hold for $m = k$. Then,

$$\begin{aligned}
\| \Delta_{k+1} \|_\infty^2 &\leqslant A \int_{t_0}^{t} | \Delta_k |^2 \, d\tau + B \int_{t_0}^{t} | \Delta_k |^2 \, dF(\tau) \\
&= (\| \Delta_0 \|_\infty^2 / k!)\Big(A \int_{t_0}^{t} \{A(\tau - t_0) + B[F(\tau) - F(t_0)]\}^k \, d\tau \\
&\quad + B \int_{t_0}^{t} \{A(\tau - t_0) + B[F(\tau) - F(t_0)]\}^k \, dF(\tau)\Big) \\
&= (\| \Delta_0 \|_\infty^2 / k!)\Big(A \Big\{t[A(t - t_0) + B(F(t) - F(t_0))]^k \\
&\quad - k \int_{t_0}^{t} \{A(\tau - t_0) + B[F(\tau) - F(t_0)]\}^{k-1} \, \tau d[A(\tau - t_0) \\
&\quad + B(F(\tau) - F(t_0))]\Big\} + B \Big\{F(t)[A(t - t_0) + B(F(t) - F(t_0))]^k \\
&\quad - k \int_{t_0}^{t} F(\tau)\{A(\tau - t_0) + B[F(\tau) - F(t_0)]\}^{k-1} \, d[A(\tau - t_0) \\
&\quad + B(F(\tau) - F(t_0))]\Big\}\Big) \\
&= (\| \Delta_0 \|_\infty^2 / k!)\Big(\{A(t - t_0) + B[F(t) - F(t_0)]\}^k \{At + BF(t)\} \\
&\quad - k \int_{t_0}^{t} \{A(\tau - t_0) + B[F(\tau) - F(t_0)]\}^k \, d[A(\tau - t_0) \\
&\quad + B(F(\tau) - F(t_0))] \\
&\quad - k[At_0 + BF(t_0)] \int_{t_0}^{t} \{A(\tau - t_0) + B[F(\tau) - F(t_0)]\}^{k-1} \\
&\quad \times d[A(\tau - t_0) + B(F(\tau) - F(t_0))]\Big) \\
&= (\| \Delta_0 \|_\infty^2 / k!)(\{A(t - t_0) + B[F(t) - F(t_0)]\}^{k+1} \{1 - [k/(k + 1)]\}) \\
&\leqslant [\| \Delta_0 \|_\infty^2 / (k + 1)!](AL + BV)^{k+1}
\end{aligned} \tag{29}$$

Thus we obtain (28) for all m. Now, given λ, $0 < \lambda < 1$, we can choose a sufficiently large n such that $(AL \times BV)^n / n! \leqslant \lambda^2$, and therefore

$$\| \varphi^n \xi - \varphi^n \eta \|_\infty = \| \Delta n \|_\infty \leqslant \lambda \, \| \Delta_0 \|_\infty = \lambda \, \| \xi - \eta \|_\infty \tag{30}$$

Hence φ^n is a contraction, and this proves the lemma.

Theorem 2.1. Let conditions 1–3 of Lemmas 2.1 and 2.2 hold. Then, there exists a unique solution process $x(t)$ satisfying (11) for every $t \in [t_0, T]$.

Proof. It follows from Lemma 2.2 that for some n, the power φ^n of the mapping $\varphi: L_\infty([t_0, T]) \to L_\infty([t_0, T)]$ defined by (16) is a contraction. Therefore, it follows (cf. Kolmogorov and Fomin,[7] p. 70) that the mapping φ has a unique fixed point; that is, there is a unique (up to equivalence) stochastic process $x(t)$ such that $\varphi x(t) = x(t)$. Thus Eq. (15), and hence Eq. (11), is satisfied by $x(t)$. This proves the theorem.

The following theorem concerns continuous solutions of Eq. (11).

Theorem 2.2. Let $F(t)$ be continuous, and let $C([t_0, T])$ be the Banach space of continuous processes. Define φ on $C[t_0, T])$ by (16). Let the hypotheses of Theorem 2.1 hold. Then ,there exists a unique (up to equivalence) continuous solution $x(t)$ satisfying Eq. (2.1) for every $t \in [t_0, T]$.

Proof. φ is defined on $C([t_0, T])$. Since $F(t)$ is continuous, from (19)–(21), it follows that the range of φ is contained in $C([t_0, T])$; that is, $\varphi: C([t_0, T]) \to C([t_0, T])$. The assertion of the theorem now follows if we replace L_∞ by its subspace $C([t_0, T])$ in Lemmas 2.1 and 2.2 and Theorem 2.1.

3. MOMENTS OF THE SOLUTION PROCESS

For the sake of convenience, we shall consider the motion of a free particle; that is, we assume that $m(\cdot, \cdot) \equiv 0$. In this case, the motion is represented by the equation

$$x(t) = x_0 - \int_{t_0}^{t} \int_{t_0}^{s} \gamma(s-u)\, x(u)\, du\, ds + \int_{t_0}^{t} \sigma(s, x(s))\, dM(s) \tag{31}$$

In this section, we study the moments of the solution process generated by Eq. (31). First, we compute the mean and variance of the solution process.

From (31) and the fact that $\mathscr{E}\{\int_{t_0}^{T} \xi(t)\, dM(t)\} = 0$ (cf. Skorohod[3]),

$$\mathscr{E}\{x(t) - x_0\} = - \int_{t_0}^{t} \int_{t_0}^{s} \gamma(s-u)\, \mathscr{E}\{x(u)\}\, du\, ds \tag{32}$$

Now,

$$[x(t) - x_0]^2 = \left[\int_{t_0}^{t} \int_{t_0}^{s} \gamma(s-u)\, x(u)\, du\, ds\right]^2 + \left[\int_{t_0}^{t} \sigma(s, x(s))\, dM(s)\right]^2$$
$$- 2 \int_{t_0}^{t} \int_{t_0}^{s} \gamma(s-u)\, x(u)\, du\, ds \int_{t_0}^{t} \sigma(s, x(s))\, dM(s)$$

so that

$$\mathscr{E}\{[x(t) - x_0]^2\} = \mathscr{E}\left\{\left[\int_{t_0}^{t} \int_{t_0}^{s} \gamma(s - u)\, x(u)\, du\, ds\right]^2\right\} + \int_{t_0}^{t} \mathscr{E}\{[\sigma(s, x(s))]^2\}\, dF(s) \tag{33}$$

To obtain (33), we have used the fact that

$$\mathscr{E}\left\{\left|\int_{t_0}^{T} \xi(t)\, dM(t)\right|^2\right\} = \int_{t_0}^{T} \mathscr{E}\{|\, \xi(t)|^2\}\, dF(t)$$

Fubini's theorem, and the fact that

$$\mathscr{E}\left\{\int_{t_0}^{T} \xi(t)\, dM(t)\right\} = 0$$

We denote by $\mathscr{D}^2\{x\}$ the variance of the random variable X. Then,

$$\begin{aligned} \mathscr{D}^2\{x(t) - x_0\} &= \mathscr{E}\{[x(t) - x_0]^2\} - \mathscr{E}\{[x(t) - x_0]^2\} \\ &= \mathscr{D}^2\left[\int_{t_0}^{t} \int_{t_0}^{s} \gamma(s - u)\, x(u)\, du\, ds\right] \\ &\quad + \int_{t_0}^{t} \mathscr{E}[\sigma(s, x(s))]^2\, dF(s) \end{aligned} \tag{34}$$

Without any appeal to mechanics, one can easily see that Eq. (4) holds for any differentiable stochastic process. So, let us next consider the moment equations for the solution process generated by Eq. (31). Let $\varphi(t, x)$ be an arbitrary function with bounded continuous derivatives φ_x, φ_{xx}, etc. Using Taylor's theorem, we have

$$\Delta\varphi = \varphi_x \Delta x + \tfrac{1}{2}\varphi_{xx}(\Delta x)^2 + o(\Delta x)^2 \tag{35}$$

From (31), we have

$$\mathscr{E}\{x(t + \Delta t) - x(t) |\, x(t)\} = -\left[\int_{t_0}^{t} \gamma(t - s)\, x(s)\, ds\right] \Delta t + o(\Delta t) \tag{36}$$

and

$$\mathscr{E}\{[x(t + \Delta t) - x(t)]^2 \mid x(t)\} = \sigma^2(t, x(t))[F(t + \Delta t) - F(t)] + o(\Delta t) \tag{37}$$

Noting that $\varphi(t, x(t))$ is $\mathscr{F}_t$-measurable, we have

$$\mathscr{E}\{\Delta\varphi \mid x\} = -\left[\varphi_x(t, x(t)) \int_{t_0}^{t} \gamma(t - x)\, x(s)\, ds\right] \Delta t + \tfrac{1}{2}\varphi_{xx}(t, x(t))\, \sigma^2(t, x(t))\, \Delta F(t) + o(\Delta t) \tag{38}$$

Taking expectations on both sides of (38),

$$\mathscr{E}\{\Delta\varphi\} = -\mathscr{E}\left\{\varphi_x(t, x)\int_{t_0}^{t}\gamma(t-s)\,x(s)\,ds\right\}\Delta t$$
$$+\tfrac{1}{2}\mathscr{E}\{\varphi_{xx}(t, x)\,\sigma^2(t, x(t))\}\,\Delta F(t) + o(\Delta t) \tag{39}$$

Dividing throughout by Δt and taking the limit as $\Delta t \to 0$, we obtain

$$(d/dt)\,\mathscr{E}\{\varphi(t, x)\} = -\int_{t_0}^{t}\gamma(t-s)\,\mathscr{E}\{\varphi_x(t, x(t))\,x(s)\}\,ds$$
$$+\tfrac{1}{2}\mathscr{E}\{\varphi_{xx}(t, x(t))\,\sigma^2(t, x(t))\}(d/dt)\,F(t) \tag{40}$$

We obtain the moment equations for the solution process from (40) by taking $\varphi(t, x)$ equal to x, x^2, etc. For example,

$$(d/dt)\,\mathscr{E}\{x\} = -\int_{t_0}^{t}\gamma(t-s)\,\mathscr{E}\{x(s)\}\,ds \tag{41}$$

and

$$(d/dt)\,\mathscr{E}\{x^2(t)\} = -2\int_{t_0}^{t}\gamma(t-s)\,\mathscr{E}\{x(t)\,x(s)\}\,ds + \mathscr{E}\{\sigma^2(t, x(t))\}(d/dt)\,F(t) \tag{42}$$

Since the Itô–Doob integral, as a function of the upper limit, is a martingale, the family

$$\{x(t) - x_0 + \int_{t_0}^{t}\int_{t_0}^{s}\gamma(s-u)\,x(u)\,du\,ds,\ \mathscr{F}_t\,,\ t\in[t_0\,, T]\}$$

is a martingale. Here, without much loss of generality, we shall assume that the initial velocity x_0 is a positive constant. Then, from standard martingale inequalities, we obtain the following inequalities:

$$\mu\Big(\sup_{t\in[t_0, T]}\Big|\,x(t) + \int_{t_0}^{t}\int_{t_0}^{s}\gamma(s-u)\,x(u)\,du\,ds\,\Big| \geqslant x_0\Big)$$
$$\leqslant x_0^{-1}\mathscr{E}\left\{\Big|\,x(T) + \int_{t_0}^{T}\int_{t_0}^{t}\gamma(t-s)\,x(s)\,ds\,dt\,\Big|\right\} \tag{43}$$

and

$$\mathscr{E}\left\{\sup_{t\in[t_0, T]}\left\{x(t) - x_0 + \int_{t_0}^{t}\int_{t_0}^{s}\gamma(s-u)\,x(u)\,du\,ds\right\}^2\right\}$$
$$\leqslant 4\mathscr{E}\left\{\left[x(T) - x_0 + \int_{t_0}^{T}\int_{t_0}^{s}\gamma(s-u)\,x(u)\,du\,ds\right]^2\right\}$$
$$\leqslant 8\left(\mathscr{E}\{[x(T) - x_0]^2\} + \mathscr{E}\left\{\left[\int_{t_0}^{T}\int_{t_0}^{s}\gamma(s-u)\,x(u)\,du\,ds\right]^2\right\}\right) \tag{44}$$

Finally, we consider the asymptotic behavior of $\mathscr{D}^2\{x(t)\}$. Let us assume that $\sigma(t, x(t)) \geqslant \theta > 0$, and that $F(t)$ monotonically increases indefinitely as $t \uparrow \infty$. In this case, we have

$$\begin{aligned}\mathscr{D}^2\{x(t)\} &\geqslant \mathscr{D}^2\{x(t) - x_0\} \\ &\geqslant \int_{t_0}^{t} \mathscr{E}\{[\sigma(t, x(s))]^2\}\, dF(s) \\ &\geqslant \theta^2[F(t) - F(t_0)] \to \infty \qquad \text{as} \quad t \uparrow \infty\end{aligned}$$

Also, the variance $\mathscr{D}^2\{x(t)\} \to \infty$ as $t \uparrow \infty$ in the case of slow relaxation since, in this case, the particle becomes a Brownian particle and Eq. (4) reduces to the classical Langevin equation.

4. A COMPARISON THEOREM FOR SOLUTION PROCESSES

For a free particle, the integral equation describing the motion is given by

$$\begin{aligned}x(t) &= x_0 - \int_{t_0}^{t} \int_{t_0}^{s} \gamma(s - u)\, x(u)\, du\, ds + \int_{t_0}^{t} \sigma(s, x(s))\, dM(s) \\ &= x_0 - \int_{t_0}^{t} (\gamma * x)(s)\, ds + \int_{t_0}^{t} \sigma(s, x(s))\, dM(s) \qquad (45)\end{aligned}$$

In this section, we shall prove a comparison theorem for the solution process which roughly says that, under suitable assumptions, if two particles start with the same initial velocity, the motion of the particle that moves under a greater damping force (e.g., in a fluid with more viscosity) is slower than the other particle. In other words, if damping is increased, the motion slows down.

For $i = 1,2$, let $\{x_i(t),\ t \in [t_0, T]\}$ be the continuous solution process of the equation

$$x_i(t) = x_0^{(i)} - \int_{t_0}^{t} (\gamma_i * x_i)(s)\, ds + \int_{t_0}^{t} \sigma(s, x_i(s))\, dM(s) \qquad (46)$$

with the same $\sigma(\cdot, \cdot)$ and $M(\cdot)$.

Theorem 4.1. Let γ_1, γ_2, and $\sigma(\cdot, \cdot)$ satisfy the following conditions:

(1) $\sigma(t, x)$ is continuous in its variables $t \in [t_0, T]$ and $x \in (-\infty, \infty)$.

(2) $\sigma(t, x) > 0$, and for every $c > 0$, there exists an $\alpha > 1/2$ and $K > 0$ such that for $|x| \leqslant c$, $|y| \leqslant c$,

$$|\sigma(t, x) - \sigma(t, y)| \leqslant K\, |x - y|^{\alpha} \qquad (47)$$

(3) $x_1(t)$ and $x_2(t)$ are the continuous solutions of Eq. (46) under the conditions of Theorem 2.2.

(4) $\gamma_1 * x_1 < \gamma_2 * x_2$ for every $t \in [t_0, T]$.

(5) Let τ be a stopping time.

(6) $x_1(\tau) = x_2(\tau)$ a.s.

Then, there is a τ_1 with $\tau_1 > \tau$ a.s., and for $s \in (\tau, \tau_1)$, the inequality $x_1(s) > x_2(s)$ is satisfied a.s.

Proof. For $s \in [t_0, T]$, define $\psi(s) = 1$ if and only if

(i) $\tau \leqslant s$

and

(ii) $$\inf_{\tau \leqslant u \leqslant s} [(\gamma_2 * x_2)(u) - (\gamma_1 * x_1)(u)] > \tfrac{1}{2}[(\gamma_2 * x_2)(\tau) - (\gamma_1 * x_1)(\tau)]$$

and $\psi(s) = 0$ if and only if

(iii) $s < \tau$

and (iv) for $s > \tau$, we have

$$\inf_{\tau \leqslant u \leqslant s} [(\gamma_2 * x_2)(u) - (\gamma_1 * x_1)(u)] \leqslant \tfrac{1}{2}[(\gamma_2 * x_2)(\tau) - (\gamma_1 * x_1)(\tau)]$$

Also define for $k > 0$, $c > 0$, and $s \in [t_0, T]$

$$\psi_k{}^c(s) = I_{[0,c]}\{\sup_{t_0 \leqslant u \leqslant s} [|x_1(u)| + |x_2(u)|]\} I_{[\tau,\tau+k]}\psi(s) \tag{48}$$

We need the following lemma for the proof of Theorem 4.1.,

Lemma 4.1. The following holds with probability 1:

$$\lim_{k \to 0} k^{-1} \int_{t_0}^{T} \psi_k{}^c(s)[\sigma(s, x_1(s)) - \sigma(s, x_2(s))]\, dM(s) = 0 \tag{49}$$

Proof of Lemma. First, we obtain the following estimate:

$$\mathscr{E}\left\{\int_{t_0}^{T} \psi_k{}^c(s)[\sigma(s, x_1(s)) - \sigma(s, x_2(s))]\, dM(s)\right\}^2$$

$$= \int_{t_0}^{T} \mathscr{E}\{\psi_k{}^c(s)[\sigma(s, x_1(s)) - \sigma(s, x_2(s))]^2\}\, dF(s)$$

{since $\mathscr{E}\left\{\left|\int_{t_0}^T \xi(t)\, dM(t)\right|^2\right\} = \int_{t_0}^T \mathscr{E}\{|\xi(t)|^2\}\, dF(t)$, and $[\psi_k{}^c(s)]^2 = \psi_k{}^c(s)]$

$$\leqslant K^2 \int_{t_0}^T \mathscr{E}\{\psi_k{}^c(s)|\, x_1(s) - x_2(s)|^{2\alpha}\}\, dF(s$$

[by Eq. (47)]

$$\leqslant K^2\mathscr{E}\left\{\left[\int_{t_0}^T \psi_k{}^c(s)\, dF(s)\right]^{1-\alpha} \left[\int_{t_0}^T \psi_k{}^c(s)|\, x_1(s) - x_2(s)|^2\, dF(s)\right]^{\alpha}\right\}$$

(by Holder's inequality)

$$\leqslant K^2 \left[\int_{t_0}^T \mathscr{E}\{\psi_k{}^c(s)\}\, dF(s)\right]^{1-\alpha} \left[\int_{t_0}^T \mathscr{E}\{\psi_k{}^c(s)|\, x_1(s) - x_2(s)|^2\}\, dF(s)\right]^{\alpha}$$

(by Holder's inequality)

$$\leqslant K^2[\mathscr{E}\{F(\tau + k) - F(\tau)\}]^{1-\alpha} \left[\int_{t_0}^T \mathscr{E}\{\psi_k{}^c(s)|\, x_1(s) - x_2(s)|^2\}\, dF(s)\right]^{\alpha} \tag{50}$$

In the above estimation, we have also used Fubini's theorem several times. We note that $\psi_k{}^c(s) = 1$ if and only if

(v) $\quad \sup_{t_0 \leqslant u \leqslant s} \{|\, x_1(u)| + |\, x_2(u)|\} \leqslant c$

(vi) $\quad s \in [t_0\,, T] \cap [\tau, \tau + k]$

and

(vii) $\quad \psi(s) = 1$

If $u \leqslant s$, then $\psi_k{}^c(s) = 1$ implies $\psi_k{}^c(u) = I_{[\tau,s]}(u)$. From this,

$$\psi_k{}^c(s)[x_1(s) - x_2(s)] = \psi_k{}^c(s) \int_\tau^s \psi_k{}^c(u)[(\gamma_2 * x_2)(u) - (\gamma_1 * x_1)(u)]\, du$$
$$+ \psi_k{}^c(s) \int_\tau^s \psi_k{}^c(u)[\sigma(u, x_1(u)) - \sigma(u, x_2(u))]\, dM(u) \tag{51}$$

From the uniform continuity of $(\gamma * x)$ on $[t_0\,, T]$, there is an H such that

$$I_{[0,c]}(x)(|\, \gamma_1 * x_1| + |\, \gamma_2 * x_2|) \leqslant H \tag{52}$$

We now have

$$\mathscr{E}\left\{\int_{t_0}^{T} \psi_k{}^c(s)[\sigma(s, x_1(s)) - \sigma(s, x_2(s))]\, dM(s)\right\}^2$$

$$\leqslant K^2[\mathscr{E}\{F(\tau+k) - F(\tau)\}]^{1-\alpha} \left[\int_{t_0}^{T} \mathscr{E}\{\psi_k{}^c(s)|\, x_1(s) - x_2(s)|^2\}\, dF(s)\right]^{\alpha}$$

[by (50)]

$$\leqslant K^2[\mathscr{E}\{F(\tau+k) - F(\tau)\}]^{1-\alpha} \left\{\int_{t_0}^{T} \mathscr{E}\left\{\psi_k{}^c(s)\Big[8H^2k^2\right.\right.$$

$$\left.\left.+ 2\left(\int_{t_0}^{s} \psi_k{}^c(u)[\sigma(u, x_1(u)) - \sigma(u, x_2(u))]\, dM(u)\right)^2\right\}\Big]\, dF(s)\right\}^{\alpha}$$

[by 51) and (52)]

$$\leqslant K^2[\mathscr{E}\{F(\tau+k) - F(\tau)\}]^{1-\alpha} \Big[8H^2k^2[\mathscr{E}\{F(\tau+k) - F(\tau)\}]$$

$$+ 2\int_{t_0}^{T} \mathscr{E}\left\{\psi_k{}^c(s)\left(\int_{t_0}^{s} \psi_k{}^c(u)[\sigma(u, x_i(u)) - \sigma(u, x_2(u))]\, dM(u)\right)^2\right\}$$

$$\times\, dF(s)\Big]^{\alpha}$$

$$\leqslant K_1k^{2\alpha}\mathscr{E}\{F(\tau+k) - F(\tau)\} + K_2[\mathscr{E}\{F(\tau+k) - F(\tau)\}]^{1-\alpha}$$

$$\times \left[\int_{t_0}^{T} \mathscr{E}\left\{\psi_k{}^c(s)\left(\int_{t_0}^{s} \psi_k{}^c(u)[\sigma(u, x_1(u)) - \sigma(u, x_2(u))]\, dM(u)\right)^2\right\}\right.$$

$$\left.\times\, dF(s)\right]^{\alpha} \tag{53}$$

where K_1 and K_2 are constants. But

$$\int_{t_0}^{T} \psi_k{}^c(s)\left\{\int_{t_0}^{s} \psi_k{}^c(u)[\sigma(u, x_1(u)) - \sigma(u, x_2(u))]\, dM(u)\right\}^2 dF(s)$$

$$\leqslant [F(\tau+k) - F(\tau)] \max_s \left[\int_{t_0}^{s} \psi_k{}^c(u)\{\sigma(u, x_1(u)) - \sigma(u, x_2(u))\}\, dM(u)\right]^2$$

Using the martingale inequality for this estimation, we obtain

$$\mathscr{E}\left\{\int_{t_0}^{T} \psi_k{}^c(s)\left(\int_{t_0}^{s} \psi_k{}^c(u)[\sigma(u, x_1(u)) - \sigma(u, x_2(u))]\, dM(u)\right)^2 dF(s)\right\}$$

$$\leqslant 4\mathscr{E}\{F(\tau+k) - F(\tau)\}\int_{t_0}^{T} \mathscr{E}\{\psi_k{}^c(u)[\sigma(u, x_1(u)) - \sigma(u, x_2(u))]^2\}\, dF(s) \tag{54}$$

Put

$$\nu(k) = \mathscr{E}\left\{\int_{t_0}^{T} \psi_k{}^c(u)[\sigma(u, x_1(u)) - \sigma(u, x_2(u))]\, dM(u)\right\}^2$$

From (53), (54), and the stochastic integral isometry, for some constant c_0 and c_1, the following inequality obtains:

$$\nu(k) \leqslant c_0 k^{2\alpha}[\mathscr{E}\{F(\tau + k) - F(\tau)\}] + c_1 \mathscr{E}\{F(\tau + k) - F(\tau)\}[\nu(k)]^\alpha$$

Hence,

$$\begin{aligned} &\nu(k)[\mathscr{E}\{F(\tau + k) - F(\tau)\}]^{-1}k^{-2\alpha} \\ &\quad \leqslant c_0 + c_1 k^{-2\alpha+2\alpha^2}[\mathscr{E}\{F(\tau + k) - F(\tau)\}]^\alpha \\ &\qquad \times (\nu(k)[\mathscr{E}\{F(\tau + k) - F(\tau)\}]^{-1}k^{-2\alpha})^\alpha \end{aligned} \tag{55}$$

We claim that there are positive constants D and δ such that

$$\nu(k)[\mathscr{E}\{F(\tau + k) - F(\tau)\}]^{-1}k^{-2\alpha} \leqslant D \qquad \text{if} \quad 0 < k \leqslant \delta \tag{56}$$

We prove this by contradiction. Suppose that (56) is not true and that there is a sequence $\{k_i\}$ with the properties that $k_i \to 0$ and that

$$\nu(k_i)[\mathscr{E}\{F(\tau + k_i) - F(\tau)\}]^{-1}k^{-2\alpha} \to \infty \qquad \text{as} \quad i \to \infty$$

Let $B = [t_0, T] \times [-c, c]$ and $\frac{1}{2} < \beta < \alpha < 1$. By hypothesis, $\sigma(\cdot, \cdot)$ is bounded, say by the constant $\tilde{N}$, and if $(t, x), (t, y) \in B$, we have by (47) that

$$\begin{aligned} |\sigma(t, x) - \sigma(t, y)| &\leqslant K|x - y|^\alpha \\ &\leqslant K|x - y|^\beta \qquad \text{if} \quad |x - y| \leqslant 1 \\ |\sigma(t, x) - \sigma(t, y)| &\leqslant 2\tilde{N} \\ &\leqslant 2\tilde{N}|x - y|^\beta \qquad \text{if} \quad |x - y| > 1 \end{aligned}$$

Thus, if $N_0 = \max\{K, 2\tilde{N}\}$, we have

$$|\sigma(t, x) - \sigma(t, y)| \leqslant N_0 |x - y|^\beta$$

From (55), taking $k_i = k$, we have

$$\begin{aligned} &\{\nu(k)[\mathscr{E}\{F(\tau + k) - F(\tau)\}]^{-1}k^{-2\alpha}\}^{1-\alpha} \\ &\quad \leqslant c_0(\nu(k)[\mathscr{E}\{F(\tau + k) - F(\tau)\}]^{-1}k^{-2\alpha})^{-\alpha} \\ &\qquad + c_1[\mathscr{E}\{F(\tau + k) - F(\tau)\}k^{-1}]^\alpha k^{\alpha(2\alpha-1)} \end{aligned}$$

Since $\alpha < 1$ and $[\nu(k_i)[\mathscr{E}\{F(\tau + k_i) - F(\tau)\}]^{-1}k_i^{-2\alpha}] \to \infty$, the left-hand side of the above inequality goes to ∞ as $i \to \infty$. But $\alpha > \frac{1}{2}$ and the first term on the right goes to zero. The second term also vanishes by the monotonicity of F. Thus we get the contradiction $\infty \leqslant 0$. This establishes our claim (56). Hence

$$\mathscr{E}\left\{\int_{t_0}^{T} \psi_k{}^c(s)[\sigma(s, x_1(s)) - \sigma(s, x_2(s))]\, dM(s)\right\}^2 \leqslant D\mathscr{E}\{F(\tau + k) - F(\tau)\}\, k^{-2\alpha}$$

Let

$$\eta(s) = I_{[0,c]}\{\sup_{t_0 \leqslant u \leqslant s} [|\, x_1(u)| + |\, x_2(u)|\}\, \psi(s)[\sigma(u, x_1(u)) - \sigma(u, x_2(u))]$$

and

$$\xi(t) = \int_{t_0}^{t} \eta(s)\, dM(s)$$

Clearly, $\{\xi(t),\, t \in [t_0\,, T]\}$ is a martingale and hence $\{\xi(\tau + k)\}$ is a martingale. This implies that $\{\lambda(k) = [\xi(\tau + k) - \xi(\tau)],\, k \in R^+\}$ is a martingale. Therefore, from the martingale inequality we have, for $0 < k_0 < \delta$,

$$\mathscr{E}\left\{\sup_{0 < k < k_0} \left(\int_{t_0}^{T} \psi_k{}^c(s)[\sigma(s, x_1(s)) - \sigma(s, x_2(s))]\, dM(s)\right)^2\right\}$$
$$\leqslant 4D\mathscr{E}\{F(\tau + k_0) - F(\tau)\}\, k^{-2\alpha} \tag{57}$$

Consequently,

$$\mu\left\{\sup_{2^{-\eta-1} \leqslant k \leqslant 2^{-2}} \left| k^{-1} \int_{t_0}^{T} \psi_k{}^c(s)[\sigma(s, x_1(s)) - \sigma(s, x_2(s))]\, dM(s)\right| > n^{-1}\right\}$$
$$\leqslant \mu\left\{\sup_{0 \leqslant k \leqslant 2^{-2}} \left|\int_{t_0}^{T} \psi_k{}^c(s)[\sigma(s, x_1(s)) - \sigma(s, x_2(s))]\, dM(s)\right| \leqslant n^{-1}2^{-n-1}\right\}$$
$$\leqslant n^2 2^{2n+1}\mathscr{E}\left\{\sup_{0 \leqslant k \leqslant 2^{-2}} \left|\int_{t_0}^{T} \psi_k{}^c(s)[\sigma(s, x_1(s)) - \sigma(s, x_2(s))]\, dM(s)\right|\right\}^2$$

(by Chebyshev's inequality)

$$\leqslant n^2 2^{2n+2} 4D\mathscr{E}\{F(\tau + 2^{-n}) - F(\tau)\}(2^{-n})^{2\alpha}$$
$$= 16D2^{-n(2\alpha-2)}n^2\mathscr{E}\{F(\tau + 2^{-n}) - F(\tau)\} \tag{58}$$

We now make a further assumption on F. Let us assume that F is continuously differentiable (F is monotone). One can avoid the above assumption by a random time change on the continuous martingale $M(t)$. From the continuity assumption on $F'(u)$, we have a bounded derivative, say $F'(u) \leqslant K_0$. Also,

$F(\tau + 2^{-n}) - F(\tau) = 2^{-n}F'(u)$ for some $u \in [\tau, \tau + 2^{-n}]$. Under the said assumptions on F,

$$[\text{Eq. (58)}] \leqslant 16Dk_0n^2 2^{-n(2\alpha-1)} \tag{59}$$

Since $\alpha > \frac{1}{2}$, $\sum_{n=1}^{\infty} n^2 2^{-n(2\alpha-1)} < \infty$. Therefore, by the Borel–Cantelli lemma, it follows almost surely that

$$\lim_{k\to 0} k^{-1} \int_{t_0}^{T} \psi_k{}^c(s)[\sigma(s, x_1(s)) - \sigma(s, x_2(s))]\, dM(s) = 0$$

This completes the proof of the lemma.

Proof of Theorem 4.1. From the definition of $\psi_k{}^c(s)$, we have

$$\begin{aligned}
&\psi_k{}^c(\tau + k) \int_{t_0}^{T} \psi_k{}^c(u)[(\gamma_2 * x_2)(u) - (\gamma_1 * x_1)(u)]\, du \\
&\quad \geqslant \psi_k{}^c(\tau + k) \int_{t_0}^{T} \psi_k{}^c(u)\, 2^{-1}[(\gamma_2 * x_2)(\tau) - (\gamma_1 * x_1)(\tau)]\, du \\
&\quad = 2^{-1}k\psi_k{}^c(\tau + k)[(\gamma_2 * x_2)(\tau) - (\gamma_1 * x_1)(\tau)]
\end{aligned}$$

Therefore

$$\begin{aligned}
&\psi_k{}^c(\tau + k)[x_1(\tau + k) - x_2(\tau + k)] \\
&\quad = \psi_k{}^c(\tau + k) \int_{t_0}^{T} \psi_k{}^c(u)[(\gamma_2 * x_2)(u) - (\gamma_1 * x_1)(u)]\, du \\
&\qquad + \psi_k{}^c(\tau + k) \int_{t_0}^{T} \psi_k{}^c(u)[\sigma(u, x_1(u)) - \sigma(u, x_2(u))]\, dM(u) \\
&\quad \geqslant k\psi_k{}^c(\tau + k)\Big\{2^{-1}[(\gamma_2 * x_2)(\tau) - (\gamma_1 * x_1)(\tau)] \\
&\qquad + k^{-1} \int_{t_0}^{T} \psi_k{}^c(u)[\sigma(u, x_1(u)) - \sigma(u, x_2(u))]\, dM(u)\Big\}
\end{aligned} \tag{60}$$

By Lemma 4.1, there exists a positive h such that, for $0 < k < h$,

$$\begin{aligned}
&k^{-1} \left| \int_{t_0}^{T} \psi_k{}^c(u)[\sigma(u, x_1(u)) - \sigma(u, x_2(u))]\, dM(u) \right| \\
&\quad < 4^{-1}[(\gamma_2 * x_2)(\tau) - (\gamma_1 * x_1)(\tau)]
\end{aligned}$$

so that, for $0 < k < h$,

$$\psi_k^c(\tau + k)[x_1(\tau + k) - x_2(\tau + k)] \geqslant \tfrac{1}{4}\psi_k^c(\tau + h)[(\gamma_2 * x_2)(\tau) - (\gamma_1 * x_1)(\tau)] \tag{61}$$

But for almost all ω, there are positive constants c and k_0 such that $\psi_k^c(\tau + k) = 1$ for $0 < k < k_0$. Hence the theorem.

Theorem 4.2. Let conditions 1–4 of Theorem 4.1 hold. If

$$\mu\{x_1(t_0) = x_2(t_0)\} = 1$$

then $x_1(t) \geqslant x_2(t)$ a.s.

Proof. Let

$$\tau = \inf\{t \geqslant t_0 : x_1(t, \omega) \geqslant x_2(t, \omega)\}$$

τ is a stopping time. By the continuity of the solution, we have $x_1(\tau(\omega), \omega) = x_2(\tau(\omega), \omega)$. By Theorem 4.1, there is a $k_0 > 0$ such that $x_1(t) > x_2(t)$ for $\tau < t < \tau + k_0$. Thus if τ_1 is the first zero of $(x_1(t) - x_2(t))$, then τ_1 is a stopping time and there is a stopping time τ_2 such that $x_1(t) > x_2(t)$ for $t \in (\tau_1, \tau_2)$. Continuing this process, we observe that a zero of $(x_1(t) - x_2(t))$ follows a zero of the same. Arranging these zeros in an increasing sequence τ_i, if there is a maximal zero τ^*, then by Theorem 4.1, there is a positive h such that $x_1(t) = x_2(t)$ for $t \in (\tau^*, \tau^* + h)$. The difference $x_1(t) - x_2(t)$ retains its sign in (τ^*, T) so that $x_1(t) > x_2(t)$ for (τ^*, T). Thus for all $t \in [t_0, T]$, $x_1(t) > x_2(t)$ almost surely.

5. POLYGONAL APPROXIMATION TO THE SOLUTION

In this section, we show, by adapting cauchy's method, that the solution of

$$x(t) = x_0 + \int_{t_0}^{t} \int_{t_0}^{s} \gamma(s - u)\, x(u)\, du + \int_{t_0}^{t} m(u, x(u))\, du + \int_{t_0}^{t} \sigma(u, x(u))\, dM(u) \tag{62}$$

can be approximated by a Cauchy polygon, and this will give an estimate of the accuracy of the approximation. Standard texts on differential equations (cf. Birkhoff and Rota[8]) give the Cauchy polygonal method and Gronwall's lemma, which is used in this section.

From the uniform continuity of $(\gamma * x)(s)$ on $[t_0, T]$, there is a constant $K_0 > 0$ such that $|\gamma * x| \leqslant K_0$. We shall assume that $m(t, x)$ and $\sigma(t, x)$ are jointly continuous for $t \in [t_0, T]$ and $x \in (-\infty, \infty)$ and satisfy the usual Lipschitz continuity in x. Let $x(t)$ be bounded by $A > 0$ in mean square.

For fixed ξ, $m(t, \xi)$ and $\sigma(t, \xi)$ are continuous on $[t_0, T]$ and hence are bounded there. By the Lipschitzian continuity, there exists a constant $L \geqslant K_0$ such that

$$\| m(t, x)\|_2 \leqslant L \qquad \text{and} \qquad \| \sigma(t, x)\|_2 \leqslant L$$

Under the assumptions made above, we have

$$g(\delta) \to 0 \qquad \text{as} \quad \delta \to 0 \tag{63}$$

where

$$g(\delta) = \max[\Gamma(\delta), m(\delta), \sigma(\delta)]$$

and

$$\begin{aligned}
\Gamma(\delta) &= \sup_{|t-s|<\delta} \|(\gamma * x)(t) - (\gamma * x)(s)\|_2 \\
m(\delta) &= \sup_{|t-s|<\delta} \| m(t, x(t)) - m(s, x(s))\|_2 \\
\sigma(\delta) &= \sup_{|t-s|<\delta} \| \sigma(t, x(t)) - \sigma(s, x(s))\|_2
\end{aligned}$$

The function $F(t)$ associated with the martingale $M(t)$ is continuous and monotone. We also assume that $F(t)$ is differentiable everywhere and has a bounded derivative $f(t)$, with bound B.

Let π: $t_0 < t_1 < \cdots < t_n = T$ be a partition of $[t_0, T]$ and let $\| \pi \|$ denote the mesh of the partition π, i.e., $\| \pi \| = \max\{(t_i - t_{i-1}): 1 \leqslant i \leqslant n\}$. We want to show that for each $\epsilon > 0$, there is a $\delta > 0$ with the following property: Let $x(t)$ be a solution process of Eq. (62) and π be a partition with $\| \pi \| < \delta$. If $x_\pi(t)$ is the Cauchy polygonal approximate solution of Eq. (62), then,

$$\| x_\pi(t) - x(t)\|_2 < \epsilon, \qquad t_0 \leqslant t \leqslant T \tag{64}$$

Now, we define the Cauchy polygon as follows:

(i) $\quad \mathrm{X}_\pi(t_0) = x_0$

and (ii) if x_π has been defined for $t_0 \leqslant t \leqslant t_i$, we define

$$\begin{aligned}
x_\pi(t_{i+1}) = x_\pi(t_i) &+ (\gamma * x_\pi)(t_i)(t_{i+1} - t_i) \\
&+ m(t_i, x_\pi)(t_{i+1} - t_i) + \sigma(t_i, x_\pi)[M(t_{i+1}) - M(t_i)]
\end{aligned}$$

and for $t \in (t_i, t_{i+1})$, we define x_π by linear interpolation between $x_\pi(t_i)$ and $x_\pi(t_{i+1})$.

First, we consider $\| x(t) - x(s)\|_2$:

$$\| x(t) - x(s)\|_2^2 = \mathscr{E}\{x(t) - x(s)\}^2$$

$$= \mathscr{E}\left\{\int_s^t (\gamma * x)(u)\, du + \int_s^t m(u, x(u))\, du + \int_s^t \sigma(u, x(u))\, dM(u)\right\}^2$$

$$\leqslant 3\mathscr{E}\left\{\int_s^t (\gamma * x)(u)\, du\right\}^2 + 3\mathscr{E}\left\{\int_s^t m(u, x(u))\, du\right\}^2$$

$$+ 3\mathscr{E}\left\{\int_s^t \sigma(u, x)\, dM(u)\right\}^2 \tag{65}$$

For $t_0 \leqslant s < t \leqslant T$, we have

$$\mathscr{E}\left\{\int_s^t (\gamma * x)(u)\, du\right\}^2 \leqslant L^2(T - t_0)(t - s)$$

$$\mathscr{E}\left\{\int_s^t m(u, x(u))\, du\right\}^2 \leqslant L^2(T - t_0)(t - s)$$

$$\mathscr{E}\left\{\int_s^t \sigma(u, x(u))\, dM(u)\right\}^2 \leqslant L^2B(t - s)$$

Using these estimates in (65), we obtain

$$\| x(t) - x(s)\|_2^2 \leqslant 3L^2[2(T - t_0) + B](t - s)$$

Therefore there is a constant $\alpha > 0$ such that

$$\| x(t) - x(s)\|_2 \leqslant \alpha(t - s)^{1/2} \tag{66}$$

We formalize the definition of x_π as follows: For any $t \in [t_0, T]$, there is an i, $0 \leqslant i \leqslant n - 1$, such that $t \in [t_i, t_{i+1})$ and we define

$$x_\pi(t) = x_0 + \int_{t_0}^t (\gamma * x_\pi)(T_\pi s)\, ds + \int_{t_0}^t m(T_\pi s, x_\pi)\, ds$$

$$+ \int_{t_0}^{t_i} \sigma(T_\pi s, s_\pi)\, dM(s) + q(t) \int_{t_0}^{t_{i+1}} \sigma(t_i, x_\pi)\, dM(s) \tag{67}$$

where

$$T_\pi t = t_i \qquad \text{if} \quad t_i \leqslant t \leqslant t_{i+1}, \quad 0 \leqslant i \leqslant n - 1$$

and

$$q(t) = (t - t_i)/(t_{i+1} - t_i)$$

Now, from (67),

$$
\begin{aligned}
x_\pi(t) - x(t) &= \int_{t_0}^{t} [(\gamma * x_\pi)(T_\pi s) - (\gamma * x)(T_\pi s)]\, sd \\
&+ \int_{t_0}^{t} [(\gamma * x)(T_\pi s) - (\gamma * x)(s)]\, ds \\
&+ \int_{t_0}^{t} [m(T_\pi s, x_\pi) - m(T_\pi s, x)]\, ds \\
&+ \int_{t_0}^{t} [m(T_\pi s, x) - m(s, x)]\, ds \\
&+ \int_{t_0}^{t_i} [\sigma(T_\pi s, x_\pi) - \sigma(T_\pi s, x)]\, dM(s) \\
&+ \int_{t_0}^{t_i} [\sigma(T_\pi s, x) - \sigma(s, x)]\, dM(s) \\
&+ q(t) \int_{t_0}^{t_{i+1}} \sigma(t_i\, , x(t_i))\, dM(s) + \int_{t_0}^{t} -\sigma(s, x(s))\, dM(s) \\
&= \sum_{k=1}^{9} I_k \qquad (68)
\end{aligned}
$$

where I_k is the kth integral appearing in the expression (68). Therefore

$$
\mathscr{E}\{x_\pi(t) - x(t)\}^2 \leqslant 9 \sum_{k=1}^{9} \mathscr{E}\{I_k{}^2\} \qquad (69)
$$

Define $\varphi(t)$ as follows: for $t \in [t_0\, , T)$,

$$
\varphi(t) = \sup_{s \in [t_0, t]} \mathscr{E}\{[x_\pi(s) - x(s)]^2\} \qquad (70)
$$

Next, we shall estimate $\| I_k \|_2^2$, $1 \leqslant k \leqslant 9$. We will use Schwarz's inequality in several places. Applying the Cauchy–Schwarz inequality for the integrals $\int_{t_0}^{t}$ and $\int_{t_0}^{T_\pi s}$, and noting that $(\gamma^2 * 1)$, where 1 is the function identically equal to 1, can be majorized by a constant $c > 0$, we get

$$
\mathscr{E}\{I_1{}^2\} \leqslant c \int_{t_0}^{t} \varphi(s)\, ds
$$

$$
\mathscr{E}\{I_2{}^2\} \leqslant (t - t_0)^2\, [\Gamma(\delta)]^2
$$

and

$$\mathscr{E}\{I_3^2\} \leqslant (t - t_0)\, K^2 \int_{t_0}^{t} \mathscr{E}\{x_\pi - x\}^2\, ds$$

$$\leqslant K^2(t - t_0) \int_{t_0}^{t} \varphi(s)\, ds$$

using the Cauchy–Schwarz inequality and the Lipschitz condition,

$$\mathscr{E}\{I_4^2\} \leqslant (t - t_0)^2[m(\delta)]^2$$

Using the stochastic integral isometry and the Lipschitz condition, we obtain

$$\mathscr{E}\{I_5^2\} \leqslant K^2 B \int_{t_0}^{t_i} \varphi(s)\, ds, \qquad \mathscr{E}\{I_6^2\} \leqslant (t_i - t_0)\, B[\sigma(\delta)]^2$$

$$\mathscr{E}\{I_7^2\} = [q(t)]^2 \int_{t_0}^{t_{i+1}} \mathscr{E}\{\sigma(t_i\,, x_\pi) - \sigma(t_i\,, x)\}^2\, dF(s)$$

$$\leqslant [q(t)]^2\, K^2 \int_{t_i}^{t_{i+1}} \varphi(t_i)\, dF(s) \leqslant K^2 B \int_{t_i}^{t} \varphi(t_i)\, ds$$

$$\mathscr{E}\{I_8^2\} \leqslant L^2 B \,\|\, \pi \,\|\,, \qquad \mathscr{E}\{I_9^2\} \leqslant L^2 B \,\|\, \pi \,\|$$

We denote by $\tilde{L}$ the $\max\{K, L\}$. In the following, we combine the estimates for $\| I_5 \|_2^2$ and $\| I_7 \|_2^2$. From the above estimates, we obtain

$$\varphi(t) \leqslant c \int_{t_0}^{t} \varphi(s)\, ds + (t - t_0)^2\, [g(\delta)]^2$$

$$+ \tilde{L}^2(t - t_0) \int_{t_0}^{t} \varphi(s)\, ds + (t - t_0)^2\, [g(\delta)]^2$$

$$+ \tilde{L}^2 B \int_{t_0}^{t} \varphi(s)\, ds + (t - t_0)\, B[g(\delta)]^2 + 2B\tilde{L}^2 \,\|\, \pi \,\|$$

$$= \psi(t) + \xi(t) \int_{t_0}^{t} \varphi(s)\, ds \tag{71}$$

where

$$\psi(t) = \{[g(\delta)]^2[2(t - t_0)^2 + B(t - t_0)] + 2B\tilde{L}^2\|\, \pi \,\|\}$$

and

$$\xi(t) = [c + \tilde{L}^2\{B + (t - t_0)\}]$$

Applying Gronwall's lemma to (71), we obtain

$$\varphi(t) \leqslant \psi(t) + \int_{t_0}^{t} \xi(t)\, \psi(s) \left[\exp \int_{s}^{t} \xi(u)\, du\right] ds \tag{72}$$

Inequality (72) yields (64) upon noting that

$$\psi(T) = \max_t \psi(t) \to 0 \qquad \text{as} \quad \| \pi \| \to 0$$

For a given admissible error $\tilde{\epsilon} > 0$ and the Cauchy polygonal approximate solution x_π, clearly the L_2-norm of $x(s)$ is majorized by the sum of the error $\tilde{\epsilon}$ and

$$\sup_{s \in [t_0, t]} \| x_\pi(s) \|_2$$

This enables us to find $\psi(t)$ and $\xi(t)$. Hence $\varphi(t)$ can be estimated, which gives the accuracy of the approximation.

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